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Xuejun Xu, Ales Janka, Jean-Antoine Desideri

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

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Xuejun Xu * , Aleš Janka , Jean-Antoine Désidéri †

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Abstract: In this report, the Cascadic Multigrid method (CMG), viewed as a "one-way" multigrid method not involving pre-smoothing or restriction steps, is defined and tested in the case of a simple 2D advection-diffusion problem. Jacobi and Gauss-Seidel smoothers are employed. While the Cascadic Multigrid method is very effective for relatively small Reynolds numbers, the convergence rate deteriorates when advection and diffusion approximately balance each other. However, for larger Reynolds numbers, the method improves somewhat and achieves a mesh-independent convergence rate. In the next step, we recommend that the CMG method should be designed and tested for boundary layers in conjunction with adaptive meshes and/or semi-coarsening.

Key-words: Cascadic multigrid, finite elements, finite volumes, advection-diffusion

* Inst. of Computat. Mathematics, Chinese Academy of Sciences, P.O.Box 2719, Beijing 100080, China
† INRIA Sophia Antipolis, 2004, route des Lucioles, BP 93, 06902 Sophia Antipolis, France

Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles, BP 93, 06902 Sophia Antipolis Cedex (France)
Téléphone : +33 4 92 38 77 77 —Télécopie : +33 4 92 38 77 65

Multigrille cascadique pour l'advection-diffusion

Résumé : Dans ce rapport, la méthode multigrille cascadique (MGC), vue comme une méthode multigrille à "sens-unique" n'incluant aucune étape de pré-lissage ou de restriction, est définie et testée dans le cas d'un problème simple 2D d'advection-diffusion. On utilise des lisseurs de Jacobi ou de Gauss-Seidel. Tandis que la méthode s'avère très efficace pour les nombres de Reynolds modérés, sa vitesse de convergence se dégrade lorsque l'advection et la diffusion s'équilibrent. Néanmoins, pour les nombres de Reynolds plus grands, la méthode s'améliore un peu et atteint une vitesse de convergence indépendante du maillage. Enfin, nous recommendons, pour les problèmes où l'advection domine, de coupler la méthode avec une procédure d'adaptation de maillage ou de semi-déraffinement.

Mots-clés : Multigrille cascadique, éléments finis, volumes finis, advection-diffusion

1 Introduction

Cascadic multigrid method (CMG) [2][13] is a new kind of multigrid method. Compared with usual multigrid methods, it requires no coarse grid corrections at all and may be viewed as a "one-way" multigrid. Another distinctive feature of this method is that more iterations on coarser levels are performed so as to obtain less iterations on finer levels. Numerical experiments [3],[4] show that this method is very effective for second order elliptic problems. Recently, the cascadic multigrid method has been successfully applied to plate bending problems in [14], to Stokes problems by Braess and Dehmen in [5] and to parabolic problems in [15].

The numerical simulation of the advection-diffusion equation is a simple linear model of fluid flows. In this report, we consider how the cascadic multigrid method can be applied to the following advection-diffusion equation

$$\begin{cases} -\epsilon\Delta u + \vec{v}\cdot\nabla u & = f \quad \text{in} \quad \Omega, \\ u & = g \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset R^2$ is a bounded domain, ϵ is the diffusion parameter and $\vec{v}$ is the convective parameter (wind), both ϵ and $\vec{v}$ constant on Ω. Cases of great interest correspond to $\epsilon \ll 1$ (advection-dominated problem).

Suppose equation (1) is discretized by some finite element or finite volume scheme which results in a linear algebraic equation

$$A_h u_h = f_h, \tag{2}$$

where h is the meshsize of discretized domain Ω. Usually, the stiffness matrix A_h in (2) is sparse and of large dimension. Here, it is also nonsymmetric and badly conditioned. Define the mesh Reynolds number as follows:

$$Re = \frac{h\|\vec{v}\|}{\epsilon}. \tag{3}$$

Here, $\|\cdot\|$ denotes Euclidean norm. If $Re \ll 1$, we say that equation (1) is diffusion dominated, otherwise it is advection dominated.

In recent years, a lot of efforts have been made to construct multigrid and other iterative methods for the solution of the discretized system (2). Some Krylov subspace methods are commonly used, like generalized minimal residual (GMRES), the biconjugate gradient (BCG), the quasi-mininal residual (QMR), the stabilized biconjugate gradient (BiCGSTAB) (cf. [1][10][11]). These algorithms are usually not very efficient unless an appropriate preconditioner is constructed to speed up the solution process. In [9],[16], Cai, Widlund and Xu used Schwarz type method to design the preconditioner and then the preconditioned systems are solved by the GMRES method. Moreover, Bramble, Leyk and Pasicak [7] used a symmetric operator as a preconditioner.

A lot of work has been done related to multigrid methods for this type of problems. In [8], [6], Bramble, Pasicak, Xu and Bramble, Kwak, Pasicak obtained uniform convergence estimates for V-cycle multigrid methods applied to nonsymmetric and indefinite problems for a variety of smoothers with one smoothing step. Most of the results cited above are under the assumption that the Reynolds number Re is relatively small, without large advection. Recently, Wang and Xu [12] presented a new kind of iterative method, i.e. the cross wind block iterative method, to deal with the advection dominated problem.

The main topic of this report is to construct and numerically analyze cascadic multigrid method for the advection-diffusion equation which may have large advection. In section 2, we will construct a cascadic multigrid algorithm for the advection-diffusion equation with Jacobi or Gauss-Seidel iteration as smoother on each level. In section 3, we present some numerical experiments to evaluate its efficiency. Finally, we draw some conclusions and remarks in section 4.

2 A cascadic multigrid algorithm

For simplicity, we assume $g|_{\partial\Omega} = 0$. There is, however, no difficulty to extend the method in this section to nonzero Dirichlet boundary conditions.

We first set some notations for later use. Let $W^{k,p}(\Omega)$ be L^p based Sobolev space with the norm

$$\|u\|_{k,p,\Omega} = (\int_\Omega \sum_{|\alpha|\le k} |D^\alpha u|^p dxdy)^{\frac{1}{p}},$$

where $\alpha = (\alpha_1, \cdots \alpha_d)$ be a multi-index. When $p = 2$, we denote $W^{k,2}(\Omega)$ by $H^k(\Omega)$ and omit the index p in the norm notation. $H^1_0(\Omega)$ consists of functions in $H^1(\Omega)$ that vanish on $\partial\Omega$.

Suppose that the domain Ω is regular and $f \in L^2(\Omega)$, so that the solution u of (1) is in $H^2(\Omega)$. As u solves (1), it also verifies

$$a(u, v) = (f, v) \quad \forall v \in H^1_0(\Omega), \tag{4}$$

where

$$\begin{aligned} a(u, v) &= \bar{a}(u, v) + b(u, v), \\ \bar{a}(u, v) &= \epsilon \int_\Omega \nabla u \cdot \nabla v dx, \\ b(u, v) &= \int_\Omega \vec{v} \cdot \nabla u v dx, \\ (f, v) &= \int_\Omega f v dx. \end{aligned}$$

We assume that (4) has a unique solution.

Assume that $\mathcal{T}_l, (l = 1, \ldots, L)$ be a quasi-uniform triangular or rectangular partition of Ω with the mesh size $h_l = h_L 2^{l-L}$. $\mathcal{T}_l$ is obtained by connecting the midpoints of the three edges of all triangles or of counteredges of all rectangles in $\mathcal{T}_{l+1}$. We assume that $\bar{\Omega} = \cap_{K\in\mathcal{T}_l}\bar{K}$.

Also, based on the primary finite element mesh, let us define a dual system of finite volume control cells $\mathcal{C}_l = \{C_i^l\}_{i=1}^{n_l}$ such that the borders of the cells are parts of line perpendicular to finite element edges they intersect (cf. Fig 1). For future purposes, let us define the set of all interfaces E_h by

$$E_h = \{e, e = \partial C_i \cap \partial C_j,\ i \neq j \text{ or } e = \partial C_i \cap \partial\Omega\}.$$

Also, for each $e \in E_h$ let us define a unit normal $\vec{n}_e$ to e such that $(\vec{v}, \vec{n}_e) > 0$. For two neighbouring cells C_i and C_j let us also define a sign function $s_{e,i}$, $e = \partial C_i \cap \partial C_j$; $s_{e,i} = 1$ if $\vec{n}_e$ is the exterior normal of ∂C_i, $s_{e,i} = -1$ otherwise.

Parallel to the finite volume function space V_l, let us define the finite volume space M_l of all functions piecewise constant on cells C_i^l. We have

$$\dim(M_l) = \dim(V_l).$$

While for the discretization of the diffusive term we use standard finite element techniques, the convective term has to be, for conservativity/stability reasons, formulated by finite volumes of upwind type. For the transition between the spaces M_l and V_l we define a *mass lumping* operator $L_h : V_l \to M_l$ by

$$L_h(\sum_i \alpha_i \phi_i^l) = \sum_i \alpha_i \chi_i^l,$$

where ϕ_i^l is the standard finite element basis function of V_l and χ_i^l is the standard finite volume basis function of M_l. By mass lumping technique and Gauss Theorem, we have

$$\begin{aligned} b(u,v) &= \int_\Omega \vec{v}\cdot\nabla u v d\Omega \approx \int_\Omega \vec{v}\cdot\nabla u L_h v d\Omega \\ &\approx \sum_{C_i^l}(L_h v)_i \int_{C_i^l} \vec{v}\cdot\nabla u d\Omega = \sum_{C_i^l}(L_h v)_i \int_{\partial C_i^l} \vec{v}\cdot\vec{n} u d\Gamma \\ &= \sum_{C_i^l}(L_h v)_i \sum_{e\in E_h\cap C_i^l} \int_e \vec{v}\cdot\vec{n}_e u d\Gamma. \end{aligned}$$

To discretize the term $\int_e \vec{v}\cdot\vec{n} u d\Gamma$ we use a standard finite volume technique with a simple upwind flux function $\Phi_e(L_h u)$. For $e \in E_h \cap C_i^l$, $e = C_j^l \cap C_i^l$ we thus have

$$\int_e \vec{v}\cdot\vec{n} u d\Gamma \approx s_{e,i}\Phi_e(L_h u),$$

where

$$\Phi_e(L_h u) = \vec{v}\cdot\vec{n}_e\,(L_h u)_U.$$

The symbol $(L_h u)_U$ means the value $(L_h u)_i$, if $\vec{n}_e$ points from C_i^l to C_j^l, $(L_h u)_U = (L_h u)_j$ otherwise.

Hence, we can write a discrete analog $a_l(u_l, v_l)$ to the bilinear form $a(u,v)$. For $u_l, v_l \in V_l$ we define the discretized bilinear form $a_l(u_l, v_l) : V_l \times V_l \to \mathbb{R}$ by

$$a_l(u_l, v_l) = \bar{a}(u_l, v_l) + b_l(u_l, v_l),$$

where

$$b_l(u_l, v_l) = \sum_{C_i^l} (L_h v)_i \sum_{e \in E_h \cap C_i^l} s_{e,i} \Phi_e(L_h u).$$

Then the finite element–finite volume approximation of (4) is to find $u_l \in V_l$ such that

$$a_l(u_l, v_l) = (f, v_l) \quad \forall v_l \in V_l. \tag{5}$$

The corresponding linear algebraic system to the problem (5) is

$$\bar{A}_l \bar{u}_l = \bar{f}_l, \tag{6}$$

where $\bar{A}_l$ is a sparse, nonsymmetric matrix, $\bar{u}_l$ is the vector formulation of the discrete solution u_l, $\bar{f}_l$ is the vector associated with f by a suitable L^2-projection. Here and throughout the report, we use v_l, w_l to denote functions in V_l while $\bar{v}_l, \bar{w}_l$ denote their vector representations.

The cascadic multigrid method for solving (6) can be defined as follows:

Cascadic multigrid algorithm:

(1). Set $\bar{u}_1^* = \bar{u}_1$.

(2). Perform iterations, $l = 2, ..., L$

$$\bar{u}_l^0 = I_{l-1}^l \bar{u}_{l-1}^*.$$
$$\bar{u}_l^{m_l} = \bar{C}_l^{m_l} \bar{u}_l.$$

(3). Set $\bar{u}_l^* = u_l^{m_l}$.

Here I_{l-1}^l denotes the prolongation from coarse to fine grids, $\bar{C}_l^{m_l}$ denotes m_l steps of basic iterations such as Jacobi, Gauss-Seidel iterations applied on level l.

Following [2], we call a cascadic multigrid *optimal* in the energy norm $\|\cdot\|_1 = \bar{a}(\cdot,\cdot)^{1/2}$ on level L, if we obtain both the accuracy

$$\|u_L - u_L^*\|_1 \approx \|u - u_L\|_1,$$

ie. that the iterative error is comparable to the approximation error, and the multigrid complexity

$$\text{amount of work} = O(n_L), \quad n_L = \dim(V_L).$$

If the multigrid complexity is $O(n_L \log(n_L)^\gamma)$, where γ is a fixed integer, the cascadic multigrid is *nearly optimal.*

From the general theory of the cascadic multigrid developed in [14], we know that in general the number of basic iterations m_l on each level must satisfy the following condition

$$m_l \geq \beta^{L-l} m_L,$$

for fixed $\beta \geq 1$, m_L is the number of iterations on the finest level. We can choose m_l as the smallest integer satisfying the above condition.

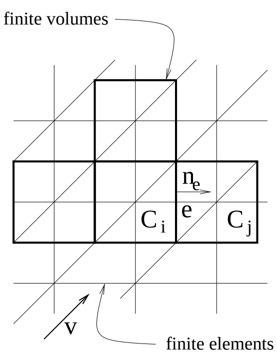

Figure 1: Linear finite element mesh and dual FVM cells

3 Numerical experiments

In this section we give some numerical resuts for solving the simple advection-diffusion problem

$$\begin{aligned} -\Delta u + \vec{v} \cdot \nabla u = 0 \quad & \text{in } \Omega = [0,1] \times [0,1] \\ u(x,0) = \cos(\pi x/2) \quad & u(x,1) = 0 \\ u(0,y) = \cos(\pi y/2) \quad & u(1,y) = 0, \end{aligned}$$

where $\vec{v} = (v_x, v_y)^T$ is the advection direction, $v_x, v_y > 0$, namely $\vec{v} = c \cdot (1,1)^T$, for some $c > 0$.

The diffusive term is discretized by linear finite element scheme, whereas the dicsretization of the convective term is done by finite volume scheme with simple upwinding of numerical fluxes (cf. Fig. 1). The FEM mesh and the corresponding FVM control volumes are structured and uniform with the mesh-size h_l.

Thus, the problem matrix $\bar{A}_L$ has the following stencil

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 - h_L v_x & 4 + h_L(v_x + v_y) & -1 \\ 0 & -1 - h_L v_y & 0 \end{bmatrix}.$$

For the solution of the discrete system, we use the cascadic multigrid as introduced in the previous sections. We use CMG with L levels and standard Jacobi (Richardson) iteration or Gauss-Seidel methods as smoothers. According to the general theory developed in [14], we choose the number of smoothing sweeps on the finest grid L as $m_L = L^2$. On the intermediate grids number $l = 1, \dots, L-1$ we apply $m_l = \beta^{(L-l)} m_L$ smoothing iterations.

The prolongation operator I_l is chosen to be a standard finite-element bilinear interpolation, ie. it has the following stencil

$$\frac{1}{4}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}.$$

The aim of our experiments is to observe the dependence of the cascadic multigrid convergence measured in energy norm (H^1-seminorm) on the mesh Reynolds number, defined by

$$Re = \|\vec{v}\| \cdot h_L.$$

3.1 Cascadic MG with Jacobi smoother

Let us have a four-level cascadic multigrid (i.e. $L = 3$) and let $\bar{C}_l^{m_l}$ be m_l steps of the standard Jacobi (Richardson) iteration

$$x_{i+1} \leftarrow (I - \bar{R}_l \bar{A}_l) x_i + \bar{R}_l \bar{b},$$

where $\bar{R}_l = 1.6 \cdot \|\bar{A}_l\|_\infty^{-1}$.

Tables 1, 2 and 3 show the energy norm of the error between the cascadic multigrid solution on the level L and the exact discrete solution on the level L as a function of the smoothing parameter β, mesh Reynolds number Re and the mesh-size h. The three tables correspond to three problems consecutively refined meshes. We can see from these tables that when $Re > 0.1$, the convergence rate of the cascadic multigrid deteriotates. Even if we increase the smoothing steps on each level, the convergence rate cannot improve quickly as we expect.

Figure 2 shows the cascadic multigrid solutions with Jacobi iteration for different mesh Re numbers and the error distributions between the cascadic multigrid solutions and the finite element and finite volume solutions. Evidently, the maximum error is concentrated in the boundary layer that forms when Re becomes large.

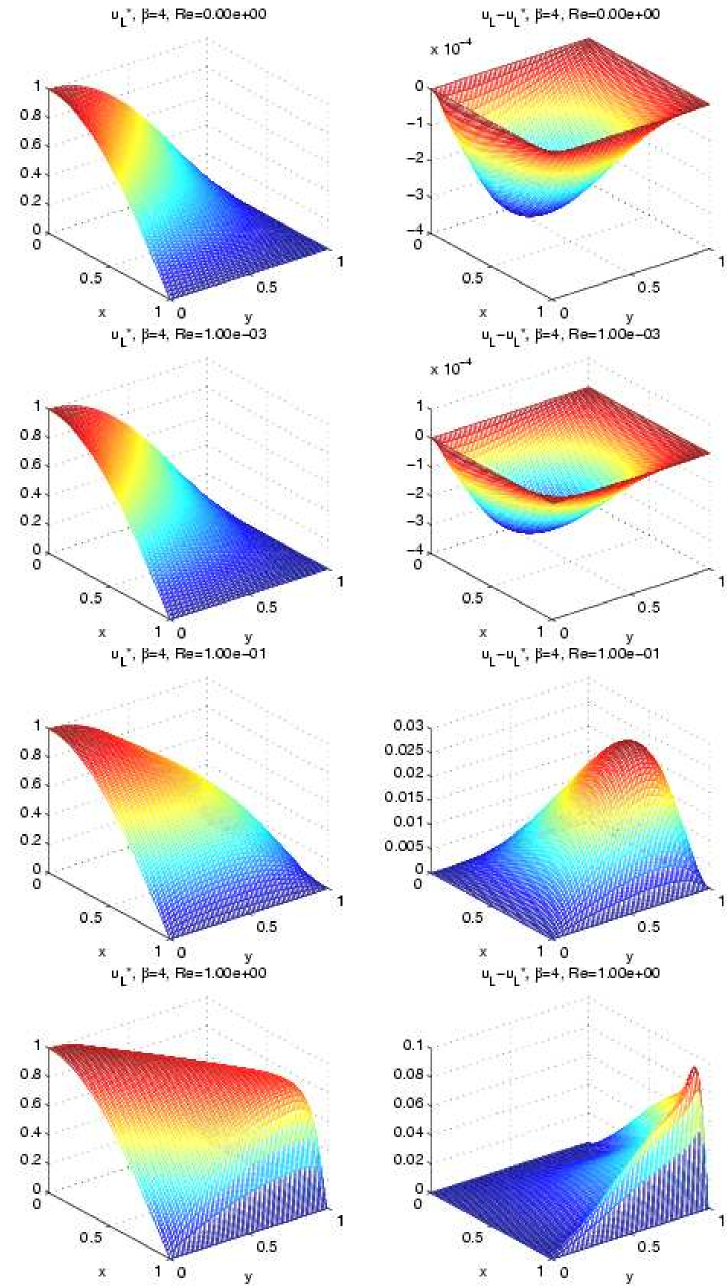

Figure 2: Cascadic MG with Jacobi smoothers: solutions and errors for different mesh Reynolds numbers *Re*

Re	Beta=4	Beta=6	Beta=8	Beta=10
0e+00	7.8956e-04	6.5106e-04	5.9323e-04	5.4694e-04
1e-06	7.8949e-04	6.5100e-04	5.9317e-04	5.4689e-04
1e-05	7.8886e-04	6.5048e-04	5.9269e-04	5.4644e-04
1e-04	7.8242e-04	6.4518e-04	5.8779e-04	5.4188e-04
1e-03	7.1004e-04	5.8500e-04	5.3172e-04	4.8946e-04
1e-02	1.3084e-03	1.1549e-03	1.0770e-03	1.0143e-03
1e-01	7.7969e-02	7.0986e-02	6.5952e-02	6.1673e-02
1e+00	4.2774e-01	3.9457e-01	3.7124e-01	3.5690e-01
1e+01	2.3817e-01	1.9827e-01	1.9776e-01	1.9776e-01

Table 1: $\|u_L - u_L^*\|_1$ for Cascadic MG with Jacobi smoother, h = 1.3889e-02

Re	Beta=4	Beta=6	Beta=8	Beta=10
0e+00	4.6326e-04	2.9998e-04	2.3251e-04	2.0925e-04
1e-06	4.6317e-04	2.9992e-04	2.3248e-04	2.0922e-04
1e-05	4.6240e-04	2.9944e-04	2.3212e-04	2.0891e-04
1e-04	4.5437e-04	2.9441e-04	2.2843e-04	2.0563e-04
1e-03	3.4552e-04	2.2487e-04	1.7714e-04	1.5970e-04
1e-02	4.3664e-03	3.2589e-03	2.7611e-03	2.5927e-03
1e-01	1.4203e-01	1.1680e-01	1.0980e-01	1.0504e-01
1e+00	5.6529e-01	5.4417e-01	5.3109e-01	5.1991e-01
1e+01	3.1349e-01	2.6570e-01	2.2429e-01	1.9507e-01

Table 2: $\|u_L - u_L^*\|_1$ for Cascadic MG with Jacobi smoother, h = 6.9444e-03

Re	Beta=4	Beta=6	Beta=8	Beta=10
0e+00	1.9508e-04	1.5418e-04	1.2196e-04	9.7851e-05
1e-06	1.9501e-04	1.5412e-04	1.2191e-04	9.7816e-05
1e-05	1.9441e-04	1.5362e-04	1.2151e-04	9.7500e-05
1e-04	1.8762e-04	1.4789e-04	1.1687e-04	9.3814e-05
1e-03	1.0291e-04	7.1470e-05	5.7937e-05	5.3028e-05
1e-02	1.0535e-02	9.1962e-03	7.8799e-03	6.7689e-03
1e-01	2.2245e-01	1.8346e-01	1.6161e-01	1.5211e-01
1e+00	7.3807e-01	7.1670e-01	7.0950e-01	7.0474e-01
1e+01	3.1730e-01	2.9895e-01	2.8090e-01	2.6334e-01

Table 3: $\|u_L - u_L^*\|_1$ for Cascadic MG with Jacobi smoother, h = 3.4722e-03

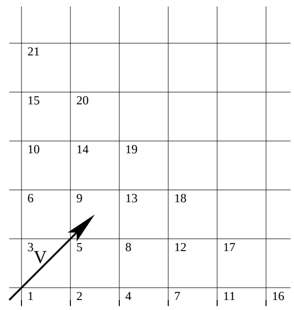

Figure 3: Numbering of nodes in Gauss-Seidel

3.2 Cascadic MG with Gauss-Seidel smoothers

Now, let us change the smoothing method $\bar{C}_l^{m_l}$ in the cascadic multigrid to Gauss-Seidel, while the other parameters are left unchanged ($L = 3$, $m_L = L^2$).

Re	Beta=4	Beta=6	Beta=8	Beta=10
0e+00	5.2204e-04	4.3680e-04	3.7652e-04	3.3223e-04
1e-06	5.2199e-04	4.3676e-04	3.7648e-04	3.3220e-04
1e-05	5.2155e-04	4.3639e-04	3.7617e-04	3.3192e-04
1e-04	5.1712e-04	4.3264e-04	3.7295e-04	3.2911e-04
1e-03	4.6575e-04	3.8890e-04	3.3525e-04	2.9620e-04
1e-02	9.7607e-04	8.5761e-04	7.7103e-04	7.0555e-04
1e-01	5.8225e-02	5.0050e-02	4.4354e-02	4.0379e-02
1e+00	1.6969e-01	1.6928e-01	1.6928e-01	1.6928e-01
1e+01	2.2060e-02	2.2060e-02	2.2060e-02	2.2060e-02

Table 4: $\|u_L - u_L^*\|_1$ for Cascadic multigrid with Gauss-Seidel smoother, $h = 1.3889\text{e-}02$

We choose the numbering of nodes in the point Gauss-Seidel method in such a way, that for advection-dominant problem Gauss-Seidel asymptotically tends to an exact solver (cf. Fig 3).

Again, Tables 4, 5 and 6 show the energy norm of the error between the cascadic multigrid solution on the level L and the exact discrete solution on the level L as a function of the smoothing parameter β, mesh Reynolds number Re and the mesh-size h. The three tables correspond to three problems with consecutively refined meshes. The convergent rates of the cascadic multigrid methods in this case are a little bit faster than the ones of cascadic multigrid methods with Jacobi smoothers, but when $Re > 0.1$, the convergence rates of the cascadic multigrid methods still deteriorates.

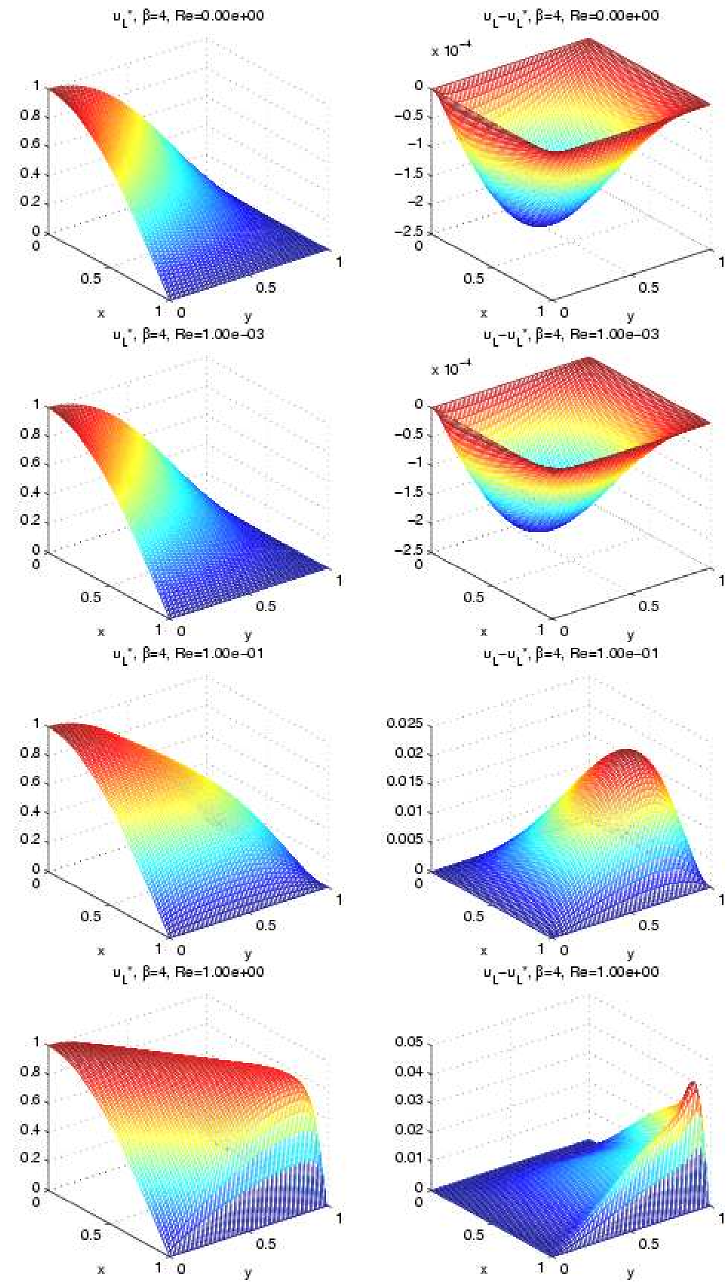

Figure 4: Cascadic MG with Gauss-Seidel smoothers: solutions and errors for different mesh Reynolds numbers Re

Re	Beta=4	Beta=6	Beta=8	Beta=10
0e+00	2.6120e-04	1.9042e-04	1.7310e-04	1.6238e-04
1e-06	2.6115e-04	1.9038e-04	1.7307e-04	1.6235e-04
1e-05	2.6073e-04	1.9009e-04	1.7280e-04	1.6209e-04
1e-04	2.5625e-04	1.8697e-04	1.6993e-04	1.5936e-04
1e-03	1.9222e-04	1.4125e-04	1.2703e-04	1.1793e-04
1e-02	2.9985e-03	2.4708e-03	2.3365e-03	2.2441e-03
1e-01	1.0166e-01	9.1907e-02	8.4924e-02	7.9384e-02
1e+00	2.2772e-01	2.1139e-01	2.0925e-01	2.0915e-01
1e+01	6.2369e-02	6.2369e-02	6.2369e-02	6.2369e-02

Table 5: $\|u_L - u_L^*\|_1$ for Cascadic multigrid with Gauss-Seidel smoothers, h = 6.9444e-03

Re	Beta=4	Beta=6	Beta=8	Beta=10
0e+00	1.4424e-04	9.9229e-05	7.3200e-05	6.0946e-05
1e-06	1.4419e-04	9.9193e-05	7.3175e-05	6.0926e-05
1e-05	1.4371e-04	9.8864e-05	7.2944e-05	6.0743e-05
1e-04	1.3823e-04	9.5012e-05	7.0200e-05	5.8549e-05
1e-03	6.2377e-05	4.8003e-05	4.4871e-05	4.3232e-05
1e-02	8.8333e-03	6.8792e-03	5.6273e-03	5.0397e-03
1e-01	1.5674e-01	1.3419e-01	1.2492e-01	1.1792e-01
1e+00	3.1401e-01	2.9634e-01	2.8200e-01	2.7257e-01
1e+01	6.8113e-02	6.8113e-02	6.8113e-02	6.8113e-02

Table 6: $\|u_L - u_L^*\|_1$ for Cascadic multigrid with Gauss-Seidel smoothers, h = 3.4722e-03

Figure 4 shows the cascadic multigrid solutions with Gauss-Seidel smoothers for different mesh Re numbers and the error distributions between the cascadic multigrid solutions and the finite element and finite volume solutions.

4 Conclusion and remarks

In this report, we have considered how a new kind of multigrid method , the Cascadic Multigrid method (CMG), can be applied to the advection-diffusion problem. We use coupled finite element and finite volume methods to approximate such kind of equation.

While the Cascadic Multigrid method is very effective for relatively small Reynolds numbers, the convergence rate deteriorates when advection and diffusion approximately balance each other. However, for larger Reynolds numbers, the method improves somewhat and achieves a mesh-independent convergence rate. In the next step, we recommend that the CMG method should be designed and tested for boundary layers in conjunction with adaptive meshes and/or semi-coarsening.

References

[1] O. Axelsson, Iterative Solution Methods, Cambridge University Press, Cambridge, 1994.

[2] F. Bornemann and P. Deuflhard, The cascadic multigrid method for elliptic problems, Numer. Math. **75**(1996), 135-152.

[3] F. Bornemann and P. Deuflhard, The cascadic multigrid method, The 8th International Conference on Domain Decomposition Methods for Partial Differential Equations, R.Glowinski, J.Periaux, Z.Shi, O.Widlund, eds., Wiley and Sons, 1997.

[4] F. Bornemann and R. Krause, Classical and Cascadic multigrid - methodogical comparison, Proceedings of the 9th International Conference onDomain Decomposition, P. Bjorstad, M. Espedal, D. Keyes, eds., John Wiley and Sons, NY, 1998.

[5] D. Braess and W. Dahmen, A cascade multigrid algorithm for the Stokes equations, Numer. Math., **82**(1999), 179-192.

[6] J. Bramble, D. Kwak, and J. Pasciak, Uniform convergence of multigrid V-cycle iterations for indefinite and nonsymmetric problems, SIAM J. Numer. Anal., **31**(1994), 1746-1763.

[7] J. Bramble, Z. Leyk, and J. Pasciak, Iterative schemes for nonsymmetric and indefinite elliptic boundary value problems,Math. Comp., **60**(1993), 1-22.

[8] J. Bramble, J. Pasciak, and J. Xu, The analysis of multigrid algorithms for nonsymmetric and indefinite elliptic problems, Math.Comp., **60** (1988), 389-414.

[9] X. Cai and O. Widlund, Domain decomposition algorithms for indefinite elliptic problems, SIAM I. Sci. Statist. Comput., **13**(1992), 243-258.

[10] W. Hackbusch, Iterative Solution of Large Sparse Systems of Equations. Springer-Verlag, Berlin, 1994.

[11] Y. Saad, Iterative Methods for Sparse Linear Systems, PWS Publishing Company, Boston, MA, 1996.

[12] F. Wang and J. Xu, A cross-wind strip block iterative method for convection-dominated problems, SIAM J. Sci. Comput., **21**(199), 646-665.

[13] V. Shaidurov, Some estimates of the rate of convergence for the cascadic conjugate gradient method, Comp. Math. Applic., **31, No 4/5** (1996), 161-171.

[14] Z. Shi and X. Xu, Cascadic multigrid method for the elliptic problem, East-Weat J. Numer. Math., **7**(1999), 199-209.

[15] Z. Shi and X. Xu, Cascadic multigrid method for parabolic problems, preprint, 1999.

[16] J. Xu, A new class of iterative methods for nonselfadjoint or indefinite problems, SIAM J. Numer. Anal., **29**(1992), 581-613.

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